

# Concentric Black Rings

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We present new supersymmetric solutions of five-dimensional minimal supergravity that describe concentric black rings with an optional black hole at the common centre. Configurations of two black rings are found which have the same conserved charges as a single rotating black hole; these black rings can have a total horizon area less than, equal to, or greater than the black hole with the same charges. A numerical investigation of these particular black ring solutions suggests that they do not have closed timelike curves.

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## I. INTRODUCTION

A remarkable feature of asymptotically flat black holes in five spacetime dimensions, in contrast to four dimensions, is that they can have event horizons with non-spherical topology. The first example of such a black hole was the discovery of the “black ring” solution of vacuum Einstein equations by Emparan and Reall [1]. This is a rotating black hole with an event horizon of topology  $S^1 \times S^2$ ; the rotation being required to prevent the ring from collapsing. The black ring of [1] carries a single rotation parameter of a possible two in five dimensions. This solution was further generalised in [2, 3, 4].

An interesting recent development is the discovery of a supersymmetric black ring solution of five-dimensional minimal supergravity [5]. The solution depends on three parameters which are uniquely specified by the mass and two independent rotation parameters,  $J_1$  and  $J_2$ . The black ring also carries electric charge which is proportional to the mass, as dictated by supersymmetry. It was shown in [5] that the infinite radius limit of this solution leads to a supersymmetric straight string solution that was found earlier in [6]. It was also shown in [5] that upon setting one of the parameters to zero, one recovers the rotating black hole solution discovered in [7, 8] (see [9] for a discussion in minimal supergravity). Recall that these black holes have an event horizon with topology  $S^3$  and carry a single rotation parameter,  $J_1 = J_2$ .

Since the black ring solution of [5] preserves supersymmetry, one suspects that more general multi-black ring solutions might exist. The supersymmetric black ring solution of [5] was found using the results of [10] which provided a classification of all supersymmetric bosonic solutions of minimal supergravity in five-dimensions. As noted in [5], the BPS equations solved by the black ring appear to be non-linear and this obscures the construc-

tion of multiple ring solutions via simple superpositions. Here we will show, also using the results of [10], that there is in fact a linear structure underlying the solutions and that this allows us to construct supersymmetric multi-ring solutions in a straightforward way. In fact we find that it is possible to superpose arbitrary numbers of concentric black rings, and moreover, it is also possible to place a black hole at the common centre. The most general solutions have  $R \times U(1)$  isometry with the  $S^1$  direction of the Killing horizons of all of the rings lying on the orbits of the  $U(1)$  Killing vector field.

We will show that there are configurations of two black rings that have exactly the same conserved charges as the rotating black hole solution of [7, 8]; this cannot be achieved with a single black ring since it has  $J_1 \neq J_2$ . We find that these two black ring configurations can have a total horizon area less than, equal to, or greater than the black hole with the same charges. In particular, and surprisingly, the rings can be entropically preferred. Moreover, a preliminary numerical investigation of these solutions suggests that these black rings do not have closed timelike curves.

Building on the original work of [11], a set of string microstates was identified in [8] which account for the rotating black hole entropy. It would be very interesting to identify the microstates corresponding to the single black ring. Presumably the entropy of the multi-black rings would then be associated with the relevant multi-string states. However, it is interesting that the conserved charges of the black holes and black rings in themselves cannot be sufficient to distinguish between different string microstates. For some discussion of black hole and black string microstate counting in a related context see [3, 4, 12].

The paper also includes a simple construction of multi-black strings, generalising the single black string solution

of [6] and we briefly mention some other generalisations of the black ring and black strings.

## II. BACKGROUND FORMALISM

The bosonic sector of five-dimensional minimal supergravity is Einstein-Maxwell theory with a Chern-Simons term [13]. A classification of the most general kinds of supersymmetric bosonic solutions of this theory was carried out in [10]. What is relevant here is the case when there exists, locally, a timelike Killing-vector  $V$  that can be constructed as a bilinear of the preserved supersymmetry parameter. For this case the line element can be written as

$$ds^2 = -f^2(dt + \omega)^2 + f^{-1}ds^2(M_4), \quad (1)$$

where  $V = \partial_t$ ,  $M_4$  is an arbitrary hyper-Kähler space, and  $f$  and  $\omega$  are a scalar and a one-form on  $M_4$ , respectively, which must satisfy

$$dG^+ = 0, \quad \Delta f^{-1} = \frac{4}{9}(G^+)^2, \quad (2)$$

where  $G^+ \equiv \frac{1}{2}f(d\omega + *\omega)$ , with  $*$  the Hodge dual on  $M_4$  and  $\Delta$  is the Laplacian on  $M_4$ . The two-form field strength is given by

$$F = \frac{\sqrt{3}}{2}d[f(dt + \omega)] - \frac{1}{\sqrt{3}}G^+. \quad (3)$$

Note that we are using the conventions of [10] but with a change in the signature of the metric.

We will only be interested in the case when  $M_4$  is flat space,  $\mathbb{R}^4$ , and it will be useful to use the following coordinates

$$\begin{aligned} ds^2(\mathbb{R}^4) &= H[dx^i dx^i] + H^{-1}(d\psi + \chi_i dx^i)^2 \\ &= H[dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\phi^2)] \\ &\quad + H^{-1}(d\psi + \cos\theta d\phi)^2 \end{aligned} \quad (4)$$

with  $H = 1/|\mathbf{x}| \equiv 1/r$  and we observe that  $\chi_i dx^i \equiv \cos\theta d\phi$  satisfies  $\nabla \times \chi = \nabla H$ . The range of the angular coordinates are  $0 < \theta < \pi$ ,  $0 < \phi < 2\pi$  and  $0 < \psi < 4\pi$ . This displays  $\mathbb{R}^4$  as a special example of a ‘‘Gibbons-Hawking’’ hyper-Kähler metric [14], and in particular we note that  $H$  is a single-centre harmonic function on the base  $\mathbb{R}^3$  with coordinates  $\mathbf{x}$ . We will further demand that the tri-holomorphic vector field  $\partial_\psi$  is a Killing vector of the five-dimensional metric. The significance of this is that the most general solution is now specified by three further harmonic functions,  $K$ ,  $L$  and  $M$  on  $\mathbb{R}^3$  [10]. In particular the general solution has

$$f^{-1} = H^{-1}K^2 + L \quad (5)$$

and if we write

$$\omega = \omega_\psi(d\psi + \cos\theta d\phi) + \hat{\omega}_i dx^i \quad (6)$$

then

$$\omega_\psi = H^{-2}K^3 + \frac{3}{2}H^{-1}KL + M \quad (7)$$

and

$$\nabla \times \hat{\omega} = H\nabla M - M\nabla H + \frac{3}{2}(K\nabla L - L\nabla K). \quad (8)$$

We also record the relation:

$$i_{\partial_\psi} G^+ = -\frac{3}{2}d(KH^{-1}) \quad (9)$$

which is useful in determining the harmonic functions given a solution not written in Gibbons-Hawking form. Finally we note here that a constant term in the harmonic function  $M$  can always be removed, locally, by a coordinate transformation that shifts  $t$ .

## III. THE SINGLE BLACK-RING SOLUTION

We now recall the solution for a single supersymmetric black-ring solution presented in [5]. We will rewrite this solution using the Gibbons-Hawking type coordinates as in (4) and extract the harmonic functions  $K$ ,  $L$  and  $M$ , which will enable us to then generalise the solution.

The solution in [5] was presented in coordinates in which the metric on  $\mathbb{R}^4$  is given by

$$\begin{aligned} ds^2(\mathbb{R}^4) &= \frac{R^2}{(x-y)^2} \left[ \frac{dy^2}{y^2-1} + (y^2-1)d\phi_2^2 \right. \\ &\quad \left. + \frac{dx^2}{1-x^2} + (1-x^2)d\phi_1^2 \right]. \end{aligned} \quad (10)$$

Note that in [5]  $\phi_1$  and  $\phi_2$  were labelled  $\phi$  and  $\psi$  respectively. The coordinates have ranges  $-1 \leq x \leq 1$  and  $-\infty < y \leq -1$ , and  $\phi_i$  have period  $2\pi$ . Asymptotic infinity lies at  $x \rightarrow y \rightarrow -1$ . Note that the apparent singularities at  $y = -1$  and  $x = \pm 1$  are merely coordinate singularities. The orientation is  $\epsilon_{y\phi_2 x \phi_1} \equiv +R^4/(x-y)^4$ . After the coordinate transformation [5]

$$\rho \sin \Theta = \frac{R\sqrt{y^2-1}}{x-y}, \quad \rho \cos \Theta = \frac{R\sqrt{1-x^2}}{x-y} \quad (11)$$

we get

$$ds^2(\mathbb{R}^4) = d\rho^2 + \rho^2(d\Theta^2 + \cos^2\Theta d\phi_1^2 + \sin^2\Theta d\phi_2^2) \quad (12)$$

and the additional coordinate transformation

$$\begin{aligned} \phi_1 &= \frac{1}{2}(\psi + \phi), & \phi_2 &= \frac{1}{2}(\psi - \phi) \\ \Theta &= \frac{1}{2}\theta, & \rho &= 2\sqrt{r} \end{aligned} \quad (13)$$

brings the metric into the Gibbons-Hawking form (4).

The single black ring solution of [5] has

$$f^{-1} = 1 + \frac{Q - q^2}{2R^2}(x-y) - \frac{q^2}{4R^2}(x^2 - y^2) \quad (14)$$

and  $\omega = \omega_{\phi_1}(x, y)d\phi_1 + \omega_{\phi_2}(x, y)d\phi_2$ , with

$$\begin{aligned}\omega_{\phi_1} &= -\frac{q}{8R^2}(1-x^2)[3Q - q^2(3+x+y)] , \\ \omega_{\phi_2} &= \frac{3}{2}q(1+y) + \frac{q}{8R^2}(1-y^2)[3Q - q^2(3+x+y)] .\end{aligned}\quad (15)$$

$Q$  and  $q$  are positive constants, proportional to the net charge and to the local dipole charge of the ring, respectively. We assume  $Q \geq q^2$  so that  $f^{-1} \geq 0$ . It is straightforward to see that

$$G^+ = \frac{3q}{4}(dx \wedge d\phi_1 + dy \wedge d\phi_2) . \quad (16)$$

By writing  $f$  and  $\omega$  in the Gibbons-Hawking coordinates  $(r, \theta, \phi, \psi)$  we can determine the three harmonic functions  $K, L, M$ . We find that they can all be expressed in terms of a single harmonic function  $h_1$  given by

$$h_1 = \frac{1}{|\mathbf{x} - \mathbf{x}_1|} \quad (17)$$

with a single centre on the negative  $z$ -axis:  $\mathbf{x}_1 \equiv (0, 0, -R^2/4)$ . Specifically:

$$\begin{aligned}K &= -\frac{q}{2}h_1 \\ L &= 1 + \frac{Q - q^2}{4}h_1 \\ M &= \frac{3q}{4} - \frac{3qR^2}{16}h_1\end{aligned}\quad (18)$$

with  $\mathbf{x} = \mathbf{x}_1$  a coordinate singularity corresponding to the event horizon of the black ring with topology  $S^1 \times S^2$ . The radius of the  $S^2$  is  $q/2$  and that of the  $S^1$  is  $l$  defined by

$$l \equiv \sqrt{3 \left[ \frac{(Q - q^2)^2}{4q^2} - R^2 \right]} . \quad (19)$$

It was argued in [5] that demanding that this is positive ensures that the solution is free of closed time-like curves (CTCs).

The ADM mass and angular momenta of the solution are given by [5]

$$\begin{aligned}M_{ADM} &= \frac{3\pi}{4G}Q , \\ J_1 &= \frac{\pi}{8G}q(3Q - q^2) , \\ J_2 &= \frac{\pi}{8G}q(6R^2 + 3Q - q^2) .\end{aligned}\quad (20)$$

The total electric charge  $Q$  satisfies  $M = (\sqrt{3}/2)Q$ , consistent with saturation of the BPS bound of [15].

As noted in [5], if we set  $R = 0$  then we find that the solution becomes the black hole solution of [7, 8], but we note that a new condition needs to be imposed for the absence of CTCs:  $4Q^3 > q^2(3Q - q^2)^2$ . If we further set  $3Q = q^2$  then we obtain the static black hole solution.

## A. A generalisation

One natural generalisation of the black ring solution of [5] is to leave  $H, K$ , and  $L$  unchanged and modify  $M$  as follows:

$$M = \frac{3qz}{4} - \frac{3qR^2z}{16}h_1 \quad (21)$$

for constant  $z$ . Clearly this modification leaves  $f$  unchanged. The change in  $\omega$  is easily calculated and in the  $(x, y, \phi_1, \phi_2)$  coordinates we find that  $\omega_{\phi_i} \rightarrow \omega_{\phi_i} + \delta\omega_{\phi_i}$  with  $\omega_{\phi_i}$  as in (15) and

$$\begin{aligned}\delta\omega_{\phi_1} &= \frac{3q}{4}(z-1)(1-x) \\ \delta\omega_{\phi_2} &= \frac{3q}{4}(z-1)(y+1) .\end{aligned}\quad (22)$$

We will not analyse the global structure of these solutions in detail here. However, we note that  $\partial_\psi$  remains space-like near the origin (at  $x = 1, y = -1$ ) which is necessary for the absence of CTCs. This would not have been the case for different choices of the constant term in (21). We also note that  $\omega_{\phi_2} = 0$  at  $y = -1$  and that  $\omega_{\phi_1} = 0$  at  $x = 1$ . However  $\omega_{\phi_1} \neq 0$  at  $x = -1$ , which indicates the presence of a Dirac-Misner string whose removal seems to require periodically identifying the time coordinate. Finally, it is interesting to observe that  $\mathbf{x} = \mathbf{x}_1$  is still just a coordinate singularity for these solutions: the analysis of [5], or the one we present below, goes through straightforwardly with the only essential change being that  $l$  gains a  $z$  dependence via  $R^2 \rightarrow zR^2$  in (19). Perhaps the Euclidean version of these solutions are worth studying further.

## IV. MULTI-BLACK RING SOLUTIONS.

We now come to the main results of the paper. We keep  $H = 1/r$  which means that the Gibbons-Hawking base space remains as  $\mathbb{R}^4$ . For  $K, L$  and  $M$  we consider the natural generalisation to a multi-centred ansatz

$$\begin{aligned}K &= -\frac{1}{2} \sum_{i=1}^N q_i h_i \\ L &= 1 + \frac{1}{4} \sum_{i=1}^N (Q_i - q_i^2) h_i \\ M &= \frac{3}{4} \sum_{i=1}^N q_i - \frac{3}{4} \sum_{i=1}^N q_i |\mathbf{x}_i| h_i\end{aligned}\quad (23)$$

with  $h_i = 1/|\mathbf{x} - \mathbf{x}_i|$  and  $Q_i, q_i$  are constants. It is easy to incorporate an extra  $z_i$  constant as above, but we shall not do so for simplicity. We take  $Q_i \geq q_i^2$  in order to ensure that  $f \geq 0$ . The constant term in  $M$  has been chosen to ensure that  $\partial_\psi$  remains spacelike at  $\mathbf{x} = 0$ . The solution is fully specified after solving (8), which is

not simple to do in general. We will solve this equation below when all of the poles are located on the  $z$ -axis, and argue that regular solutions without CTC's exist. It is easy, however, to determine the mass of the general solution and we find

$$M_{ADM} = \frac{3\pi}{4G} \left[ \sum_{i=1}^N (Q_i - q_i^2) + \left( \sum_{i=1}^N q_i \right)^2 \right]. \quad (24)$$

We now analyse what happens as  $\mathbf{x} \rightarrow \mathbf{x}_i$  for some fixed  $i$ . To do so we first make a rotation so that  $\mathbf{x}_i$  is at  $(0, 0, -R_i^2/4)$  and set up new spherical polar coordinates  $(\epsilon_i, \theta_i, \phi_i)$  in  $\mathbb{R}^3$  centred on  $\mathbf{x}_i$  and then consider an expansion in  $\epsilon_i$ .

After doing this, and solving (8), we find a coordinate singularity at  $\epsilon_i = 0$ . Motivated by a similar analysis in [5] we then introduce new coordinates

$$\begin{aligned} dt &= dv + \left( \frac{b_2}{\epsilon_i^2} + \frac{b_1}{\epsilon_i} \right) d\epsilon_i \\ d\psi &= d\phi'_i + 2(d\psi' + \frac{c_1}{\epsilon_i} d\epsilon_i) \\ \phi_i &= \phi'_i \end{aligned} \quad (25)$$

for constants  $b_j$  and  $c_j$ . In order to eliminate a  $1/\epsilon_i$  divergence in  $g_{\epsilon_i\psi'}$  and a  $1/\epsilon_i^2$  divergence in  $g_{\epsilon_i\epsilon_i}$  we take  $b_2 = q_i^2 l_i/8$  and  $c_1 = -q_i/(2l_i)$  where

$$l_i \equiv \sqrt{3 \left[ \frac{(Q_i - q_i^2)^2}{4q_i^2} - R_i^2 \right]}, \quad (26)$$

which we take to be positive. A  $1/\epsilon_i$  divergence in  $g_{\epsilon_i\epsilon_i}$  can be eliminated by a suitable choice for  $b_1$ , whose explicit expression is not illuminating (for the single black ring solution,  $b_1 = (2q_i^2 + Q_i)/(4l_i) + l_i(Q_i - q_i^2)/(3R_i^2)$ . Note the similarity of these constants with analogous constants appearing in [5], despite the fact that we are using different coordinates.) The metric can now be written

$$\begin{aligned} ds^2 &= -\frac{256\epsilon_i^4}{q_i^4 R_i^4} dv^2 - \frac{4}{l_i} dv d\epsilon_i + \frac{32\epsilon_i^3 \sin^2 \theta}{q_i R_i^4} dv d\phi'_i \\ &+ \frac{8\epsilon_i}{q_i} dv d\psi' + l_i^2 d\psi'^2 + \frac{q_i^2}{4} [d\theta_i^2 + \sin^2 \theta_i d\phi_i'^2] \\ &+ 2g_{\epsilon_i\phi'_i} d\epsilon_i d\phi'_i + 2g_{\epsilon_i\psi'} d\epsilon_i d\psi' + g_{\epsilon_i\epsilon_i} d\epsilon_i^2 \\ &+ 2g_{\psi'\phi'_i} d\psi' d\phi'_i + 2g_{v\theta_i} dv d\theta_i + 2g_{\psi'\theta_i} d\psi' d\theta_i \\ &+ 2g_{\epsilon_i\theta_i} d\epsilon_i d\theta_i + 2g_{\theta_i\phi'_i} d\theta_i d\phi'_i + \dots \end{aligned} \quad (27)$$

where  $g_{\epsilon_i\psi'}$  and  $g_{\epsilon_i\epsilon_i}$  are  $\mathcal{O}(\epsilon_i^0)$ ;  $g_{\psi'\phi'_i}$  and  $g_{\epsilon_i\theta_i}$  are  $\mathcal{O}(\epsilon_i)$ ;  $g_{v\theta_i}$  is  $\mathcal{O}(\epsilon_i^5)$ ;  $g_{\psi'\theta_i}$  is  $\mathcal{O}(\epsilon_i^2)$ ; and  $g_{\theta_i\phi'_i}$  is  $\mathcal{O}(\epsilon_i^4)$  whose explicit forms are unimportant for our considerations here, and the ellipsis denotes terms involving sub-leading (integer) powers of  $\epsilon_i$  in all of the metric components explicitly indicated.

The determinant of this metric is analytic in  $\epsilon_i$ . At  $\epsilon_i = 0$  it vanishes if and only if  $\sin^2 \theta_i = 0$ , which just corresponds to coordinate singularities. It follows that

the inverse metric is also analytic in  $\epsilon_i$  and hence the above coordinates define an analytic extension of our solution through the surface  $\epsilon_i = 0$ .

The supersymmetric Killing vector field  $V = \partial_v$  is null at  $\epsilon_i = 0$ . Furthermore  $V_\mu dx^\mu = -(2/l_i) d\epsilon_i$  at  $\epsilon_i = 0$ , so  $V$  is normal to the surface  $\epsilon_i = 0$ . Hence  $\epsilon_i = 0$  is a null hypersurface and a Killing horizon of  $V$ , i.e., the black ring has an event horizon which is the union of the Killing Horizons for each  $\epsilon_i = 0$ .

In the near horizon limit defined by scaling  $v \rightarrow v/\delta$ ,  $\epsilon_i \rightarrow \delta\epsilon_i$  and then taking the limit  $\delta \rightarrow 0$ , we find that the metric is locally the product of  $AdS_3$  with radius  $q_i$  and a two-sphere of radius  $q_i/2$ , in agreement with [5].

We can read off the geometry of a spatial cross-section of the horizon:

$$ds_{\text{horizon}}^2 = l_i^2 d\psi'^2 + \frac{q_i^2}{4} (d\theta_i^2 + \sin^2 \theta_i d\phi_i'^2). \quad (28)$$

We see that the horizon has geometry  $S^1 \times S^2$ , where the  $S^1$  and round  $S^2$  have radii  $l_i$  and  $q_i/2$ , respectively. This is precisely the geometry of the event horizon for a single supersymmetric black ring.

The above calculation shows that as one goes near to a pole the other poles have a sub-dominant effect. Thus, if one of the poles lies at the origin, i.e. we set one of the  $R_i$  to zero, a similar analysis should reveal a Killing horizon with spherical topology corresponding to the rotating black hole solution of [7, 8] (see [9] for a discussion of the near horizon geometry).

## A. Discussion

We have just shown that the multi-ring solutions all have Killing horizons with topology  $S^1 \times S^2$ . The solutions are invariant under the action of  $\partial_\psi$  (which equals  $(1/2)\partial_{\psi'}$ ) and the  $S^1$  direction of each horizon lies on an orbit of this vector field. To gain some insight into the solutions, it is useful to consider such orbits in  $\mathbb{R}^4$  which generically lie inside a two-plane.

In addition to the coordinates  $(r, \theta, \phi, \psi)$  and  $(\rho, \Theta, \phi_1, \phi_2)$  that we have already introduced for  $\mathbb{R}^4$  it will be useful to also introduce  $(r_1, \phi_1, r_2, \phi_2)$ , defined by

$$r_1 = \rho \cos \Theta, \quad r_2 = \rho \sin \Theta. \quad (29)$$

Then  $(r_i, \phi_i)$  each label an  $\mathbb{R}^2$  in polar coordinates and together these give  $\mathbb{R}^4$ . We first note that any point in  $\mathbb{R}^3$  labelled by  $(r_0, \theta_0, \phi_0)$ ,  $r_0 \neq 0$ , defines an  $S^1$ , parametrised by  $\psi$ , with radius  $r_0$  lying in a two-plane specified by  $(\theta_0, \phi_0)$ . Note also that all of these  $S^1$ 's are concentric, with common centre  $r = 0$ . For example the point  $\theta = \pi$  and  $r = R^2/4$ , corresponding to the single black ring solution, defines an  $S^1$  lying just in the  $(r_2, \phi_2)$  plane, while the point  $\theta = 0$  and  $r = R^2/4$  defines an  $S^1$  lying in the orthogonal  $(r_1, \phi_1)$  two-plane, both centred at the origin.

This provides us with a natural interpretation of our multi-black ring solutions: the location of the pole in

$\mathbb{R}^3$  corresponds to a different plane in  $\mathbb{R}^4$  (at asymptotic infinity) in which the  $S^1$  of the ring lies. Moreover, all of these  $S^1$ s are concentric. All of the poles lying in a given direction starting from the origin in  $\mathbb{R}^3$ , for example the negative  $z$ -axis, correspond to  $S^1$ s lying in the same two-plane. If the poles lie on a single line passing through the origin in  $\mathbb{R}^3$ , for example the  $z$ -axis, then the black rings lie in one of two orthogonal two-planes. This is consistent with the fact that, as we shall see, both  $\partial_{\phi_i}$  are Killing vectors of the solution in this case. Finally, our solutions also allow for the possibility of a single black hole being located at the common centre of all of these rings.

### B. Poles on the $z$ -axis.

We now consider the solutions with all poles located along the  $z$ -axis, where we can analyse the solutions in more detail. Consider the general solution (23) with  $\mathbf{x}_i = (0, 0, -k_i R_i^2/4)$  and  $k_i = \pm 1$ . Thus

$$h_i = (r^2 + \frac{k_i R_i^2}{2} r \cos \theta + \frac{R_i^4}{16})^{-1/2}. \quad (30)$$

We can solve (8) with  $\hat{\omega}$  only having a non-zero  $\phi$  component,  $\hat{\omega}_\phi$ , that is a function of  $r$  and  $\theta$  only. In particular, these solutions have an extra  $U(1)$  symmetry generated by  $\partial_\phi$ . Since  $\partial_\psi$  is also Killing, both  $\partial_{\phi_i}$  are Killing as claimed above.

To solve (8) we write  $\hat{\omega} = \hat{\omega}^L + \hat{\omega}^Q$  where  $\hat{\omega}^L$  is linear in the parameters  $q_i$  and independent of  $Q$  and  $\hat{\omega}^Q$  contains the dependence on  $Q$ . We find

$$\hat{\omega}^L = - \sum_{i=1}^N \frac{3q_i}{4} [1 - (r + \frac{R_i^2}{4})h_i](\cos \theta + k_i)d\phi \quad (31)$$

and

$$\begin{aligned} \hat{\omega}^Q = & -\frac{3}{64} \sum_{i < j} \frac{q_i q_j (\Lambda_j - \Lambda_i) h_i h_j}{(k_i R_i^2 - k_j R_j^2)} \left[ \frac{16}{h_i^2} + \frac{16}{h_j^2} \right. \\ & \left. - \frac{32}{h_i h_j} - (k_i R_i^2 - k_j R_j^2)^2 \right] d\phi \end{aligned} \quad (32)$$

where

$$\Lambda_i \equiv \frac{(Q_i - q_i^2)}{2q_i}. \quad (33)$$

By considering the asymptotic form of the solution we find that the angular momentum are given by

$$\begin{aligned} J_1 = & \frac{\pi}{8G} \left[ 2 \left( \sum_{i=1}^N q_i \right)^3 + 3 \left( \sum_{i=1}^N q_i \right) \sum_{j=1}^N (Q_j - q_j^2) \right. \\ & \left. - 3 \sum_{i=1}^N q_i R_i^2 (k_i - 1) \right] \\ J_2 = & J_1 + \frac{3\pi}{4G} \left( \sum_{i=1}^N q_i R_i^2 k_i \right). \end{aligned} \quad (34)$$

We would also like to check whether there are any Dirac-Misner strings that might require making periodic identifications of the time coordinate. We demand that  $\omega_{\phi_1}$  vanishes at  $\theta = \pi$ , which is  $r_1 = 0$ , where  $\phi_1$  is not well defined. Similarly we demand that  $\omega_{\phi_2}$  vanishes at  $\theta = 0$ , which is  $r_2 = 0$ . Now since  $\hat{\omega}$  only has a  $\phi$  component for poles lying on the  $z$ -axis we observe that  $\omega_{\phi_{1,2}} = \omega_\psi (1 \pm \cos \theta) + \hat{\omega}_\phi$ . Thus we demand that  $\hat{\omega} = 0$  at  $\theta = 0, \pi$ .

The expression for  $\hat{\omega}^L$  in (31) satisfies these conditions. What about  $\hat{\omega}^Q$ ? Consider two poles for simplicity. If  $\Lambda_1 \neq \Lambda_2$ , then analysing  $\hat{\omega}^Q$  at  $\theta = 0, \pi$ , we find that it is constant for values of  $z$  between the poles and vanishes otherwise. One might try to remove the constant by introducing another coordinate patch to cover this region with a shift in the time coordinate by an appropriate constant multiple of  $\phi_i$ . However, since  $\phi_i$  are periodic coordinates, this means that the time coordinate must be periodically identified. Thus we conclude that to avoid CTCs  $\Lambda_1 = \Lambda_2$ . It would be interesting to obtain a physical understanding of this constraint, and its obvious generalisation to arbitrary numbers of rings.

After imposing  $\Lambda \equiv \Lambda_1 = \Lambda_2 = \dots$ , the radii of the rings is given by

$$l_i \equiv \sqrt{3[\Lambda^2 - R_i^2]}. \quad (35)$$

We conclude that we can place the two rings anywhere on the  $z$ -axis provided that  $R_i^2 < \Lambda^2$ . It is also interesting to observe that as the location of the pole goes to larger values of  $|z|$ , i.e. as  $R_i$  increases, the circumference of the rings get uniformly smaller, perhaps contrary to one's intuition.

To ensure that there are configurations with no CTC's we also demand that the determinant of the  $(\phi, \psi)$  part of the five-dimensional metric remains positive. We have analysed this numerically and found that this can be achieved for specific values of the parameters. We leave more detailed analysis for future work.

It is interesting to ask whether there are configurations of two (say) black rings with the same charges as the black hole solution of [7, 8]. Since the black hole has  $J_1 = J_2$ , we conclude from (34) that we need to locate one pole on the positive axis and one on the negative axis,  $k_2 = -k_1$ , and in addition set  $q_1 R_1^2 = q_2 R_2^2$ . We denote the black hole parameters by  $\bar{Q}, \bar{q}$  and define  $j \equiv \bar{q}(3\bar{Q} - \bar{q}^2)/2$ , with  $\bar{Q} > 0$ . By matching the mass and the angular momentum of the black hole with the two rings we deduce that

$$\begin{aligned} \bar{Q} = & 2\Lambda(q_1 + q_2) + (q_1 + q_2)^2 \\ j = & (q_1 + q_2)^3 + 3\Lambda(q_1 + q_2)^2 + 3(q_1 R_1^2). \end{aligned} \quad (36)$$

We demand that  $\Lambda^2 > R_i^2$ . The area of the black hole event horizon is given by

$$A_{BH} = 2\pi^2 \sqrt{\bar{Q}^3 - j^2} \quad (37)$$

and we demand that  $j^2 < \bar{Q}^3$ , which ensures that the black hole has no CTCs, and is equivalent to

$$8\Lambda^3(q_1 + q_2)^3 + 3\Lambda^2(q_1 + q_2)^4 > 9q_1^2 R_1^4 + 6q_1 R_1^2(q_1 + q_2)^3 + 18\Lambda q_1 R_1^2(q_1 + q_2)^2. \quad (38)$$

This can be satisfied if we choose small enough  $q_1 R_1^2$ . Observe that if we set  $q_1 = q_2$ , then we can solve for  $q_1$  so that  $j = 0$ :

$$q_1 = \frac{-3\Lambda + \sqrt{9\Lambda^2 - 6R_1^2}}{4}. \quad (39)$$

However, we find that the solution has  $\Lambda$  and  $q_1$  having opposite signs, but they should have the same sign from (33).

It is interesting to compare the black hole area with the sum of the areas of the horizons of the two black rings:

$$\begin{aligned} A_{Rings} &= 2\pi^2(q_1^2 l_1 + q_2^2 l_2) \\ &= 2\pi^2\sqrt{3} \left( q_1^2 \sqrt{\Lambda^2 - R_1^2} + q_2^2 \sqrt{\Lambda^2 - R_2^2} \right). \end{aligned} \quad (40)$$

It is remarkable that parameters can be chosen so that these areas are equal. For example, if we take  $R_1 = R_2$  and

$$\begin{aligned} q_1 = q_2 &= \frac{1}{18(\Lambda^2 - R_2^2)} \left[ -16\Lambda^3 + 18R_2^2\Lambda \right. \\ &\quad \left. + \sqrt{256\Lambda^6 - 576\Lambda^4 R_2^2 + 405R_2^4 \Lambda^2 - 81R_2^6} \right] \end{aligned} \quad (41)$$

Furthermore (again with  $q_1 = q_2$ ,  $R_1 = R_2$ ), we can also arrange for  $A_{Rings} < A_{BH}$ ; for example  $A_{Rings} = \frac{1}{2}A_{BH}$  can be achieved by setting

$$q_2 = \frac{9R_2^4}{8\Lambda(8\Lambda^2 - 9R_2^2)}. \quad (42)$$

Lastly, we can also arrange for  $A_{Rings} > A_{BH}$ ; for example  $A_{Rings} = 2A_{BH}$  with  $R_1 = R_2$  and  $q_1 = q_2$  can be achieved with

$$\begin{aligned} q_2 &= \frac{1}{45(\Lambda^2 - R_2^2)} \left[ 36\Lambda R_2^2 - 32\Lambda^3 \right. \\ &\quad \left. + \sqrt{1024\Lambda^6 - 2304\Lambda^4 R_2^2 + 1701\Lambda^2 R_2^4 - 405R_2^6} \right] \end{aligned} \quad (43)$$

For the specific values

$$\Lambda = 3/2, \quad R_2 = 1 \quad (44)$$

we obtain  $q_1 = \frac{\sqrt{41}-6}{5}$  for  $A_{Rings} = A_{BH}$ ,  $q_1 = \frac{1}{12}$  for  $A_{Rings} = \frac{1}{2}A_{BH}$  and  $q_1 = \frac{2}{25}$  for  $A_{Rings} = 2A_{BH}$ . In all three cases,  $\Lambda^2 > R_i^2$  and  $j^2 < \bar{Q}^3$ .

A numerical investigation of these solutions suggests that they do not possess closed timelike curves. It would be interesting to verify this result analytically.

## V. MULTI-BLACK STRINGS

In the infinite radius limit the black ring solution of [5] gives rise to the black string solution of [6]. This solution can easily be constructed in Gibbons-Hawking form with an  $\mathbb{R}^4$  base with  $H = 1$ ,  $\chi = 0$  and

$$K = -\frac{q}{2r}, \quad L = 1 + \frac{Q}{r}, \quad M = -\frac{3q}{4r} \quad (45)$$

The expression for  $\omega$  is given by

$$\omega = -\left(\frac{3q}{2r} + \frac{3qQ}{4r^2} + \frac{q^3}{8r^3}\right)d\psi. \quad (46)$$

Following [6], we choose  $Q^2 \geq q^2$  to eliminate CTCs at  $r=0$ .

We note that modifying  $M \rightarrow -3qz/4r$  leads to  $\omega \rightarrow \omega + \delta\omega$  with

$$\delta\omega = \frac{3q(1-z)}{4} \left( \frac{1}{r} d\psi + \cos\theta d\phi \right) \quad (47)$$

which has Dirac-Misner strings which seem to require periodic identification of the time coordinate.

A more physical multi-string generalisation, with all  $z = 1$ , is given by  $H = 1$  and

$$\begin{aligned} K &= -\frac{1}{2} \sum_{i=1}^N q_i h_i \\ L &= 1 + \sum_{i=1}^N Q_i h_i \\ M &= -\frac{3}{4} \sum_{i=1}^N q_i h_i \end{aligned} \quad (48)$$

with  $h_i = 1/|\mathbf{x} - \mathbf{x}_i|$ . The solution is fully specified after solving (8), which can be done explicitly for special cases.

## VI. FINAL COMMENTS

Using the techniques of [10], we have found several supersymmetric generalisations of the supersymmetric black ring of [5] and the supersymmetric black string of [6]. The most interesting construction describes concentric black rings with a possible black hole at the origin. We found configurations of two black rings with the same asymptotic charges as a rotating black hole but with entropy greater than, equal to, or less than that of the black hole. We hope to improve upon our numerical investigations and prove that these black rings are indeed free of CTCs. Accounting for the entropy of the black ring solutions found here and in [5] is an important outstanding issue.

Since multi-supersymmetric black holes are known to exist [9], it is natural to ask if the solutions presented here can be generalised to give solutions with black rings

centred around each of the black holes. While this may be possible, more elaborate tools will be required to find them. The reason is simply that the multi-black hole solutions themselves cannot be constructed using the Gibbons-Hawking ansatz that we have used here, which assumed that the solution is invariant under the tri-holomorphic Killing vector field  $\partial_\psi$ . To see this recall that the multi-black hole solutions have  $\mathbb{R}^4$  for the base manifold  $M_4$  in (1) and  $f^{-1}$  a harmonic function on  $\mathbb{R}^4$  with isolated point singularities. However, after writing

$\mathbb{R}^4$  in Gibbons-Hawking form (4), we find that it is only the single black hole solution that is invariant under  $\partial_\psi$ .

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